PROM² for Girls

PRISMS Online Math Meet for Girls

PROM^2 for Girls 2022

March 12, 2022

Problem Committee:

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3 points

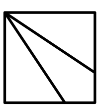
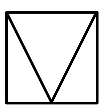
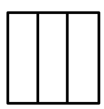
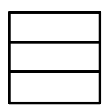
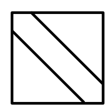
1. Evaluate $2022 - 20 \times 22$.
(A) 0 (B) 1522 (C) 1582 (D) 1622 (E) 1682

Proposed by Joseph Li
Answer: C
Solution: $2022 - 20 \times 22 = 2022 - 440 = 1582$.

2. How many times must 7 be added to -12 to get 44?
(A) 4 (B) 5 (C) 6 (D) 7 (E) 8

Proposed by Joseph Li
Answer: E
Solution: $\left(44 - (-12)\right)/7 = 56/7 = 8$.

3. Which of the following squares is not divided into three parts with equal areas?

(A)  (B)  (C)  (D)  (E) 

Proposed by Joseph Li
Answer: B
Solution:
Suppose the side length of the square is *a*. In option B, the base and the altitude of the middle triangle are both *a*. Thus, its area is $\frac{a \times a}{2} = \frac{a^2}{2}$, which is half of the area of the square.

4. How many integers between 100 and 999 (inclusive) are divisible by 37?
(A) 21 (B) 22 (C) 23 (D) 24 (E) 25

Proposed by Oliver Gao (Class of 2023)
Answer: E
Solution:

Because $2 < 100/37 < 3$ and $999/37 = 27$, the integers between 100 and 999 that are divisible by 37 are in the form of $37k$, where $3 \le k \le 27$. The answer is $27 - 3 + 1 = 25$.

5. Evaluate $\frac{331}{3} - \frac{551}{5}$.

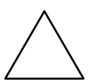
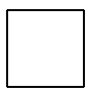
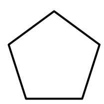
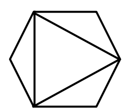
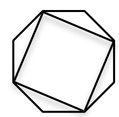
(A) 0 (B) $\frac{1}{15}$ (C) $\frac{2}{15}$ (D) $\frac{4}{15}$ (E) $\frac{8}{15}$

Proposed by Joseph Li

Answer: C

Solution: $\frac{331}{3} - \frac{551}{5} = \left(110 + \frac{1}{3}\right) - \left(110 + \frac{1}{5}\right) = \frac{1}{3} - \frac{1}{5} = \frac{2}{15}$.

6. Which of the following shapes has the most lines of symmetry?

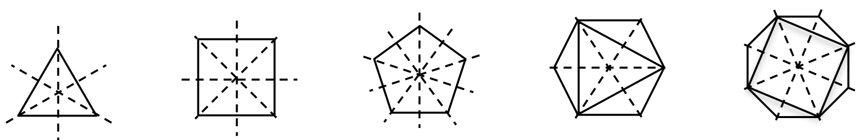
(A)  (B)  (C)  (D)  (E) 

Proposed by Joseph Li

Answer: C

Solution:

As shown in the diagrams below, the regular pentagon has a total of 5 lines of symmetry, which is the most among these options.



7. A rectangle has a width of 4 and an area of 64. What is the area of the square with the same perimeter as the rectangle?

(A) 25 (B) 100 (C) 289 (D) 400 (E) 1156

Proposed by Joseph Li

Answer: B

Solution:

The length of the rectangle is $64/4 = 16$, and its perimeter is $(16 + 4) \times 2 = 40$. Then, the side length of the square is $40/4 = 10$, and its area is $10 \times 10 = 100$.

8. On the first day of 2021, Mr. Li told all of the students at PRISMS, "2021 is the product of two consecutive prime numbers!" Which is true because $2021 = 43 \times 47$. He then proclaimed "And this is probably the only year in our lifetime that will have this property!" How many years will it take for there to be another year that is a product of two consecutive prime numbers?

(A) 94 (B) 188 (C) 282 (D) 376 (E) 470

Proposed by Joseph Li

Answer: E

Solution:

The prime number after 43 and 47 is 53, so the next year that is a product of two consecutive prime numbers is 47×53. Therefore the desired answer is

$$47 \times 53 - 43 \times 47 = 47 \times (53 - 43) = 470.$$

9. Trinity runs in a triathlon consisting of three sections: swimming, cycling, and running. She can swim at 3 miles per hour (mph), cycle at 15 mph, and run at 6 mph. If the swimming section is 1 mile long, the cycling section is 40 miles long, and the running section is 10 miles long, how many minutes will the triathlon take her in total?

(A) 216 (B) 240 (C) 263 (D) 270 (E) 280

Proposed by Christopher Qiu (Class of 2025)

Answer: E

Solution:

The number of hours the triathlon will take is

$$\frac{1}{3} + \frac{40}{15} + \frac{10}{6} = \frac{14}{3}.$$

Thus, the answer is $\frac{14}{3} \times 60 = 280$ minutes.

10. Seven children were seated at a round table, each with their own pile of candies. No two children sitting next to each other have the same number of candies. Find the minimum total number of candies that the seven children had. (Note: A pile of "candies" can be just 1 piece of candy.)

(A) 10 (B) 11 (C) 12 (D) 13 (E) 14

Proposed by Cedric Xiao (Class of 2023)

Answer: C

Solution:

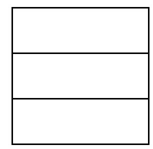
Because no two children sitting next to each other have the same number of candies, for any number k, there are at most 3 children having exactly k candies. Therefore, the minimum total number of candies is

$$1 \times 3 + 2 \times 3 + 3 \times 1 = 12.$$

4 points

11. As shown in the diagram, a square is divided into three identical rectangles. If the perimeter of each rectangle is 120, what is the perimeter of the square?

(A) 80 (B) 160 (C) 180 (D) 288 (E) 360



Proposed by Karen Miao (Class of 2023)

Answer: C

Solution:

Suppose the side length of the square is $3x$. Then the perimeter of each small rectangle is $2(3x + x) = 8x$. As given in the problem, $8x = 120$, thus $x = 15$. The perimeter of the square is $3 \times 15 \times 4 = 180$.

12. Daniella, Emma, and Valentine started packing the PROM[2] T-shirts and other materials for participants at 8:00 AM. By 8:05 AM, Daniella had packed 2 packages while Emma and Valentine had each packed 3 packages. At some time Helena joined the team and they were able to get all 300 packages packed by 10:30 AM. If Helena can pack 4 packages every 5 minutes, then what time did Helena join the team?

(A) 9:15 AM (B) 9:25 AM (C) 9:30 AM (D) 9:42 AM (E) 9:45 AM

Proposed by Joseph Li

Answer: A

Solution:

From 8:00 AM to 10:30 AM, 150 minutes have passed. In 150 minutes, the number of packages that Daniella, Emma, and Valentine packed is $150/5 \times (2 + 3 + 3) = 240$. This means Helena packed $300 - 240 = 60$ packages. The time for Helena to pack these package is $60/4 \times 5 = 75$ minutes. Therefore, Helena joined the team at 9:15 AM, which is 75 minutes before 10:30 AM.

13. The side lengths of a triangle are all prime numbers and the sum of the three lengths is 24. What is the maximum possible product of the three side lengths?

(A) 114 (B) 170 (C) 240 (D) 242 (E) 312

Proposed by Estella Xie (Class of 2023)

Answer: D

Solution:

If all three side lengths are odd, then the sum of the side lengths would be odd. But the sum, 24. is even, so either all side lengths are even or only one is even. The only even prime number is 2, so one of the side lengths must be 2. For the other two side lengths, their sum is $24 - 2 = 22,$ and the difference between them should be less than 2 (so that they can form a triangle with the side of length 2). Therefore, the other two sides must both have length 11, and the answer is $2 \times 11 \times 11 = 242$.

$$\begin{array}{r} P\ 1\ 9 \\ 4\ R\ 3 \\ +\ M\ R\ 6\ O \\ \hline 2\ 5\ M\ 6 \end{array}$$

Note: Even without using the triangular inequality, we know the product of two numbers with a fixed sum is maximized when the two numbers are equal, which will also lead us to 11+11.

14. In the equation on the right, letters '*P*', '*R*', '*O*', and '*M*' represent distinct digits among 0 to 9. What is the sum of the 4 digits that the letters represent?

(A) 14 (B) 15 (C) 16 (D) 17 (E) 18

Proposed by Lucy Wei (Class of 2023)

Answer: B

Solution:

From the unit digits, we know that $9 + 3 + O = 6$ or 16, so *O* equals 4. From the leading digit, we know *M* must represent 1 or 2, and so, we know when adding up the ten digits, there must be a carry. Because of this carry, when adding up the hundred digits, there must be a carry as well. This means *M* represents 1.

From the tens and hundreds, we have equations $1 + R + 6 + 1 = 11$ and $P + 4 + R + 1 = 15$, so $R = 3$ and $P = 7$. Therefore, the sum of the 4 digits that the letters represent is $7 + 3 + 4 + 1 = 15$.

15. In the country of PRISMagica, the Mathemagicians use three types of currency: *Matho*, *Magico*, and *Spello*. A *Magico* is worth 4 *Spellos*, and 5 *Magicos* is worth 6 *Spellos* and 7 *Mathos*. How many *Mathos* have the same value as 17 *Magicos* and 18 *Spellos*?

(A) 43 (B) 52 (C) 70 (D) 86 (E) 104

Proposed by Christopher Qiu (Class of 2025)
Answer: A
Solution:
Suppose the value of one *Spello* is 1. Then the value of one *Magico* is 4. The value of one *Mathos* is $(4 \times 5 - 6)/7 = 2$.

The value of 17 *Magicos* and 18 *Spellos* is $4 \times 17 + 18 = 86$, so it is worth $86/2 = 43$ *Mathos*.

16. There are 12 PRISMS students taking the course 'Industrial Design'. They used the 3D printers in the lab to make special badges for local kids. One of those 12 students, Angel, found that if she made 3 times as many badges as planned, then the average number of badges that each student made would be 7; and, if she made 7 times as many badges as planned, then the average number would be 13. How many badges did Angel plan to make?
(A) 10 (B) 12 (C) 15 (D) 18 (E) 20

Proposed by Joseph Li
Answer: D
Solution:
Suppose Angel plans to make *x* badges and all other 11 students plan to make *y* badges, then

$$\frac{3x+y}{12} = 7,$$
$$\frac{7x+y}{12} = 13.$$

By subtracting these two equations, we get $4x = (13-7) \times 12 = 72$, so $x = 18$.

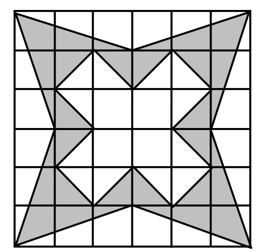
17. Felicity has 82 cents in pennies and nickels. This morning her younger brother counted the money and said it was worth $1.47. Felicity found out that her brother mistook all her nickels for dimes. How many pennies does Felicity have?
(A) 13 (B) 15 (C) 17 (D) 19 (E) 21

Proposed by Joseph Li
Answer: C
Solution:
The extra amount ($147 - 82 = 65$ cents) is due to each nickel (5 cents) being mistook as a dime (10 cents), so the number of nickels is $65/(10-5) = 13$. Felicity has $82 - 5 \times 13 = 17$ pennies.

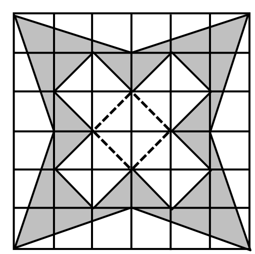
18. Lucy has 6 cards labeled from 1 to 6. She splits the cards into two piles of three. She then multiplies the sums of the cards in each pile. How many different possible products are there?
(A) 5 (B) 6 (C) 9 (D) 10 (E) 11

Proposed by Estella Xie (Class of 2023)
Answer: A
Solution:
Suppose the two sums are *a* and *b*, then $a + b = 1 + 2 + 3 + 4 + 5 + 6 = 21$. Without loss of generality, let's suppose $a < b$. The minimum possible value of *a* is $1 + 2 + 3 = 6$. Because the value of *a* can not exceed $21/2 = 10.5$, the maximum possible value of *a* is 10. The desired answer is $10 - 6 + 1 = 5$. The possible values of product *ab* are $6 \times 15, 7 \times 14, 8 \times 13, 9 \times 12, 10 \times 11$.

19. As shown on the right, 36 unit squares are assembled to form a 6 by 6 grid. What is the area of the shaded region?
(A) 14 (B) 15 (C) 16 (D) 18 (E) 19

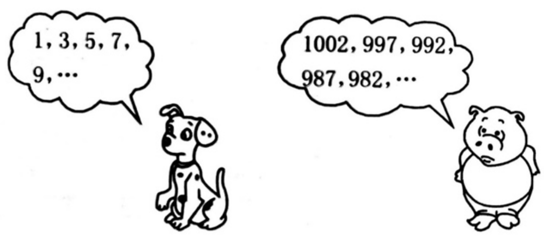


Proposed by Joseph Li
Answer: A
Solution:
Let's find the area of the unshaded regions first. The area of each unshaded triangle is $(6 \times 1)/2 = 3$. The unshaded region inside the shaded region consists of 20 small triangles and the area of each small triangle is $(1 \times 1)/2 = 0.5$. So, the total area of unshaded regions is $3 \times 4 + 0.5 \times 20 = 22$.



The area of the shaded region is $6 \times 6 - 22 = 14$.

20. The Doggy and the Piggy are counting numbers at the same speed. Which number will the Doggy and the Piggy speak out at the same time?



(A) 277 (B) 287 (C) 297 (D) 307 (E) 317

Proposed by Brian Xu (Class of 2024)
Answer: B
Solution:
Suppose the desired number is the n^{th} number they speak out. The numbers that the Doggy speaks out form an arithmetic sequence with a constant difference of 2, so its n^{th} term is $1 + 2(n-1) = 2n-1$. The numbers that the Piggy speaks out form an arithmetic sequence with a constant difference of -5, so its n^{th} term is $1002 - 5(n-1) = 1007 - 5n$.

From the equation $2n - 1 = 1007 - 5n$, we get $n = 144$. Therefore, the desired number is $2 \times 144 - 1 = 287$.

$\boxed{\text{5 points}}$

21. Five students Anna, Belle, Cinderella, Dahlia, and Elsa sat around a round table (but not necessarily in that order). Mrs. D'Angelo distributed 5 cards labeled from 1 to 5 to each student.

Anna said: The difference between the numbers held by the two students next to me is 1.

Belle said: The difference between the numbers held by the two students next to me is 2.

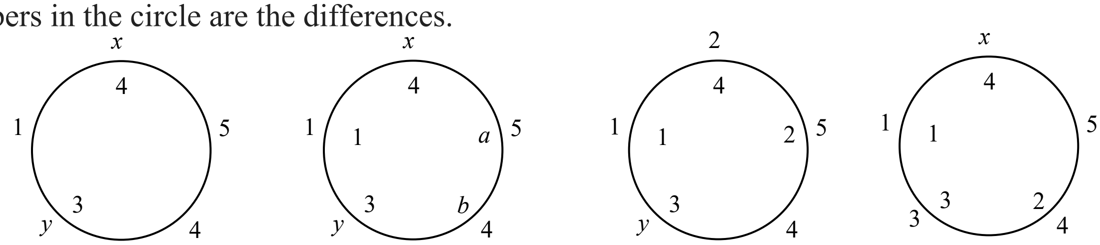
Cinderella said: The difference between the numbers held by the two students next to me is 3.

Dahlia said: The difference between the numbers held by the two students next to me is 4.

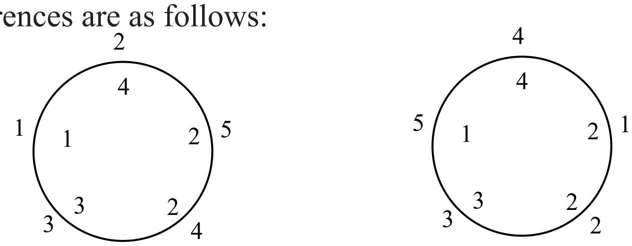
What is the difference between the numbers held by the two students next to Elsa?

(A) 1 (B) 2 (C) 3 (D) 4 (E) Cannot be determined

Proposed by Estella Xie (Class of 2023)
Answer: B
Solution:
The two numbers held by the two students next to Dahlia must be 1 and 5. Consider the two numbers next to Cinderella, they should be (1, 4) or (2, 5). Let's first consider the case 1 and 4. In the diagram below, the numbers outside the circles are the numbers on the cards and the numbers in the circle are the differences.



Because the two unknown cards x and y have numbers 2 and 3, the difference between them must be 1. Therefore Belle, who said the difference was 2, must be seated at a or b. If $a = 2$, then $4 - x = 2$ implies $x = 2$; if $b = 2$, then $5 - y = 2$ implies $y = 3$. Thus $x = 2$, $y = 3$. The cards and differences are as follows:



For the case that two numbers next to Cinderella are 2 and 5, a similar process will lead us to another possible seating above. For both cases, the desired answer is always 2.

Or

Suppose the 5 numbers are a, b, c, d, e in that order and the desired difference is x. Based on the following simple fact: the difference between two integers and the sum of those two integers have the same parity (both even or both odd), we know that $|a - c| + |b - d| + |c - e| + |d - a| + |e - b|$ and $(a + c) + (b + d) + (c + e) + (d + a) + (e + b) = 2(a + b + c + d + e)$ have the same parity. Thus $1 + 2 + 3 + 4 + x$ must be even, so x is even.

Because x is the difference between two distinct integers among 1, 2, 3, 4, and 5, the possible values of x are 2 and 4. The only two integers among 1-5 that have a difference 4 are 1 and 5, so Dahlia and Elsa can't both have difference 4. Therefore, the only possible value of x is 2.

22. $$10 + 9 + 8 + \cdots + 2 + 1 = 55$$

Karen changed some of the plus signs in the above equation to minus signs and found that the new result was 39. How many possible new equations exist?

(A) 6 (B) 7 (C) 8 (D) 21 (E) 22

Proposed by Joseph Li

Answer: A

Solution:

When Karen changes the plus sign in front of a number, let's say t, to a minus sign, the value of the expression decreases by $2t$. Therefore, the sum of the numbers that have had their signs changed is $(55 - 39)/2 = 8$.

If there is only 1 number that had its sign changed, then the number must be 8;

If there are exactly 2 numbers that had their signs changed, then they could be $1 + 7$, $2 + 6$, or $3 + 5$;

If there are exactly 3 numbers that had their signs changed, then 1 must be one of them because $2 + 3 + 4 = 9 > 8$. Therefore, the sum of the other two numbers is 7, so they could be $1 + 2 + 5$, or $1 + 3 + 4$.

It is impossible to change the signs for at least 4 numbers, because $1 + 2 + 3 + 4 = 10 > 8$. In total, there are $1 + 3 + 2 = 6$ possible new equations.

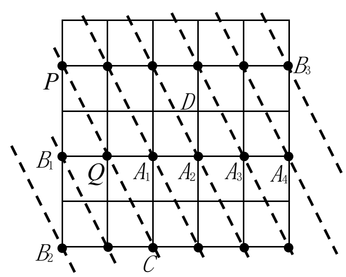
23. 25 unit squares form a 5 by 5 grid with 36 vertices. Joy wants to choose 3 points from those 36 vertices to form a triangle. She already chose *P* and *Q*. How many different vertices can the third point, *R*, be on such that the triangle *PQR* has an area that is a positive integer?

(A) 4 (B) 5 (C) 15 (D) 16 (E) 18

Proposed by Joseph Li

Answer: C

Solution:



For $1 \leq k \leq 4$, consider triangle PQA_k, its area is $2 \times |QA_k|/2 = k$, which is a positive integer. Also the area of triangle PQB_1 is $(1 \times 2)/2 = 1$, the area of triangle PQB_2 is $(4 \times 1)/2 = 2$, and the area of triangle PQB_3 is $(5 \times 2)/2 = 5$. By the property of parallel lines, when point A_k or B_k moves along a line that is parallel to *PQ*, the area of the triangle doesn't change. So all the vertices on those parallel lines can be the third point *R*, except the ones on line *PQ* (points *P*, *Q*, *C*).

Other vertices cannot be the third point. For example, point *D*. Because *D* is between the parallel lines that pass through A_2 and A_3, respectively, the area of triangle PQD is between the areas of triangle PQA_2 and triangle PQA_3, which are 2 and 3. So, the area of triangle PQD is not an integer.

Those 15 vertices on the parallel lines can be the third point.

24. Melinda and Estella go running on their school's 400-meter round track every afternoon. One day at 4:15 PM, Estella started running counterclockwise at a constant speed of 2.5 meters per second. At 4:20 PM, at the same starting point, Melinda started running clockwise at a constant speed of 3 meters per second. They both stopped running at 4:30 PM. How many times did they meet during the run?

(A) 8 (B) 9 (C) 10 (D) 11 (E) 12

Proposed by Joseph Li

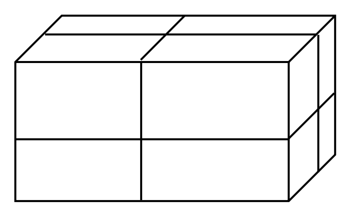
Answer: B

Solution:

When Melinda started running at 4:20 PM, Estella had already run $2.5 \times 60 \times 5 = 750$ meters. So at that moment, Estella is $400 \times 2 - 750 = 50$ meters away from Melinda, and they are running towards each other.

After 4:20 PM, when Melinda and Estella ran a sum of 50 meters, they met each other for the first time. After that, every time they ran a sum of 400 meters, they met each other again. From 4:20 to 4:30 PM, they ran a sum of $(2.5 + 3) \times 60 \times 10 = 3300$ meters. Because $(3300 - 50)/400 = 8\frac{1}{8}$, after they first met, they met each other 8 more times. Thus, in total, they met 9 times during the run.

25. There is a cuboid which has been cut three times in different directions that are parallel to the faces. After the first cut, the sum of the surface areas of the two small cuboids is 472 square inches. After the second cut, the sum of the surface areas of the four smaller cuboids is 632 square inches. After the third cut, the sum of the surface areas of the eight smaller cuboids is 752 square inches. What is the area, in square inches, of the smallest face among the six faces of the original cuboid?



(A) 48 (B) 60 (C) 80 (D) 96 (E) 120

Proposed by Joseph Li

Answer: A

Solution:

After each cut, the sum of the surface areas of the cuboids increases by 2 times the area of the cross section. Because each cut is parallel to the faces, the area of each cross section is equal to the area of the face it is parallel to in the original cuboid.

The sum of the surface areas of the cuboids after the second cut increases by $632 - 472 = 160$ square inches, so the area of one face of the original cuboid is $160/2 = 80$ square inches.

The sum of the surface areas of the cuboids after the third cut increases by $752 - 632 = 120$ square inches, so the area of one face of the original cuboid is $120/2 = 60$ square inches.

After the third cut, there is one cross section parallel to each face of the original cuboid. So, the sum of the surface areas is 2 times the surface area of the original cuboid. Thus, the surface area of the original cuboid is $752/2 = 376$ square inches. From this, we can know the area of the last face of the original cuboid is $376/2 - 80 - 60 = 48$ square inches.

26. PRISMS offers many interesting clubs to students. Christina couldn't decide which 3 clubs to choose from:

1. Choir 2. Robotics 3. Yearbook 4. Rock Band5. Theater

Each of her roommates gave her two suggestions:

Cici: "Choose the Rock Band Club" and "Don't choose the Robotics Club."

Wendy: "You should choose both the Theater Club and the Yearbook Club" and "Don't choose the Choir."

Helen: "Choose at least one from the Robotics Club and the Rock Band Club" and "Don't choose the Theater Club."

Later Christina signed up for 3 clubs and told her friends that she took exactly one suggestion from each of them. What is the sum of 3 numbers for the clubs that Christina chose?

(A) 7 (B) 8 (C) 9 (D) 10 (E) 11

Proposed by Joseph Li

Answer: E

Solution:

If Christina took the second suggestion from Cici, then she didn't choose the Rock Band Club or the Robotics Club. That means she didn't take the first suggestion from Helen; she must have taken Helen's second suggestion. Then, Christina didn't choose the Theater Club, which contradicts with the condition that she chose 3 clubs from those 5.

Therefore, Christina took the first suggestion from Cici, which means she chose both the Rock Band Club and the Robotics Club. Then she took the first suggestion from Helen and didn't take the second. So, Christina also chose the Theater Club. Now, the three clubs that Christina chose were: the Rock Band Club, the Robotics Club, and the Theater Club. It satisfies all the requirements.

27. Cameo has three buckets labeled A, B, and C that have capacities of 21 liters, 50 liters, and 60 liters, respectively. She fills half of bucket A with coffee and half of bucket B with milk. She then empties the contents of bucket A and bucket B into bucket C and fills bucket C the rest of the way up with milk and stirs it around. After stirring bucket C thoroughly, she pours 50 liters of the solution in bucket C into bucket B. In the end, what is the absolute difference between the number of liters of milk and the number of liters of coffee in bucket B?
(A) 8.5 (B) 10 (C) 29 (D) 32.5 (E) 35

Proposed by Christopher Qiu (Class of 2025)
Answer: D
Solution:
Cameo pours $21/2 = 10.5$ liters of coffee from bucket A to bucket C. After she fills bucket C, the amount of milk in bucket C is $60 - 10.5 = 49.5$ liters.

When she pours the solution in bucket C into bucket B, the ratio of coffee and milk in the solution is $10.5 : 49.5 = 7 : 33$. The answer should be $50 \times \frac{33}{40} - 50 \times \frac{7}{40} = 50 \times \frac{13}{20} = \frac{65}{2} = 32.5$ liters

28. Atticus likes to play with numbers. One day, he writes all the integers from 1 to 2022 on the whiteboard. Then, he repeatedly chooses two numbers on the whiteboard, erases them, and replaces them by their sum or product. For example, Atticus might erase 3 and 5 and then write 15 on the whiteboard. Later Atticus notices that all the remaining numbers on the whiteboard are even. Denote X as the maximum possible number of integers that remain on the whiteboard and Y as the minimum. What is the value of $X - Y$?
(A) 1010 (B) 1011 (C) 1514 (D) 1515 (E) 1516

Proposed by Cedric Xiao (Class of 2023)
Answer: D
Solution:
First we claim $Y = 1$. Our objective is to show it is possible that there is one number on the whiteboard at the end. Atticus erases 2022 and 2021, then writes 2022×2021. Next he erases 2022×2021 and 2020, then writes $2022 \times 2021 \times 2020$. Every time, he erases the largest two numbers and writes their product. In the last step, Atticus erases $2022 \times 2021 \times \cdots \times 2$ and 1, writes $2022 \times 2021 \times \cdots \times 2 \times 1$, which is an even number. Thus, $Y = 1$.

Next, Let's consider X. Every time Atticus does one operation (erases two numbers and write one), the number of integers on the whiteboard is reduced by 1. Thus, after he does n operations, there are $2022 - n$ integers on the whiteboard. If we want to find the maximum possible values for X, we just need to find the minimum possible number of operations Atticus needs to do. There are $2022/2 = 1011$ odd numbers on the whiteboard at the beginning. Every time, Atticus can erase at most 2 odd numbers. $1011/2 = 505.5$, so Atticus needs at least 506 operations to erases all the odd numbers. Therefore $X \leq 2022 - 506 = 1516$.

Last, we show $X = 1516$ is possible, meaning that Atticus does 506 operations and all the remaining numbers are even. He erases 1 and 2021, then writes the sum 2022; erases 3 and 2019, then writes the sum 2022; ⋯; erases 1009 and 1013, then writes the sum 2022. The last operation is to erase 1011 and an even number such as 2, then write the product, which is even. After these 506 operations, all the odd numbers have been erased and all the remaining 1516 integers are even. So $X = 1516$.

The desired answer $X - Y = 1516 - 1 = 1515$.

29. Kevin has a weird calculator. For any input number b, the calculator will return the value of $\frac{b-1}{b}$. For example, if Kevin inputs 2 and presses the button, the calculator will return $\frac{1}{2}$. If he pressed the button again, the calculator would return -1, because $\frac{\frac{1}{2}-1}{\frac{1}{2}} = -1$. One day Kevin inputs a number and presses the button 20 times. He records all 21 numbers in order, including the first number he inputted, and the 20 numbers returned by the calculator. He finds that the product of the last 2 numbers he recorded was 6. What is the 11th number he recorded?

(A) -6 (B) $-\frac{1}{6}$ (C) 3 (D) 6 (E) 7

Proposed by Joseph Li

Answer: E

Solution:

Suppose the first number that Kevin inputs is b. Then the calculator will return the value of $\frac{b-1}{b}$.

When Kevin presses the button again, the calculator will return the value of

$$\frac{\frac{b-1}{b}-1}{\frac{b-1}{b}} = \frac{(b-1)-b}{b-1} = -\frac{1}{b-1}.$$

When Kevin presses the button once more, the calculator will return the value of

$$\frac{-\frac{1}{b-1}-1}{-\frac{1}{b-1}} = \frac{\frac{1}{b-1}+1}{\frac{1}{b-1}} = \frac{1+(b-1)}{1} = b.$$

So the numbers that Kevin records are periodic:

$$b, \frac{b-1}{b}, -\frac{1}{b-1}, b, \frac{b-1}{b}, -\frac{1}{b-1}, \cdots$$

From the product of the last 2 numbers, we know that

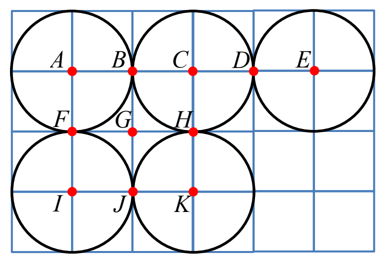
$$\frac{b-1}{b} \cdot \left(-\frac{1}{b-1}\right) = 6,$$

$$-\frac{1}{b} = 6,$$

$$b = -\frac{1}{6}.$$

Because $11/3 = 3 \cdots 2$, the answer is $\frac{b-1}{b} = \frac{-\frac{1}{6}-1}{-\frac{1}{6}} = 7$.

30. 5 circles of radius 1 are placed in a 4 by 6 grid (formed by 24 unit squares). Some vertices of the unit squares are colored red and are labeled A, B, …, K. We call a straight line *perfect* if it passes through at least two red points and divides the 5 circles into two parts with equal areas. Which of the following points does not belong to any perfect lines?



(A) A (B) E (C) F (D) J (E) K

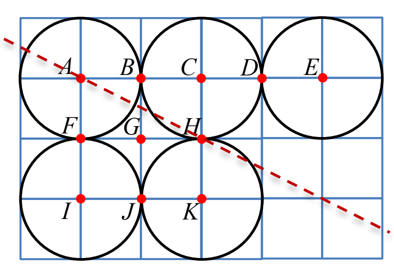
Proposed by Joseph Li

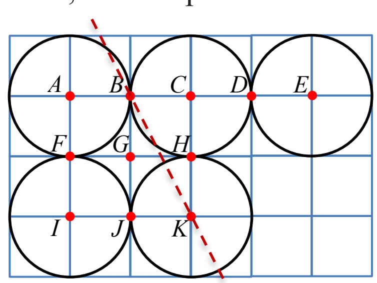
Answer: C

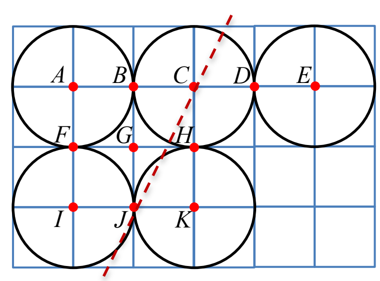
Solution:

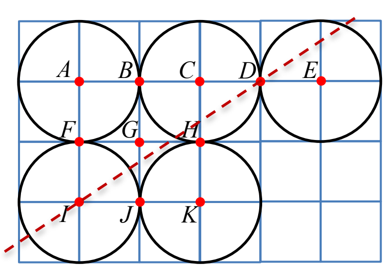
We need the following property for this problem: For a shape that is symmetrical around its center, any line that passes through its center bisects its area.

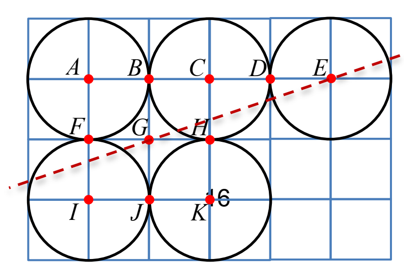
When we look at the figure consisting of 4 circles $\odot C, \odot E, \odot I$, and $\odot K$, it is symmetric over the center H. Then line AH bisects the area of those 4 circles and bisects the area of circle A because A is the center of the circle. Therefore, AH is a perfect line.

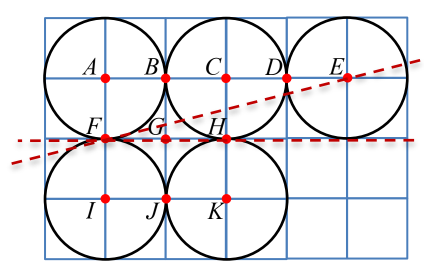












Similarly, BK is a perfect line because B is the center of the figure consisting of 2 circles $\odot A$ and $\odot C$ and K is the center of $\odot K$; CJ is a perfect line because J is the center of the figure consisting of 2 circles $\odot I$ and $\odot K$ and C is the center of $\odot C$; DI is a perfect line because D is the center of the figure consisting of 2 circles $\odot C$ and $\odot E$ and I is the center of $\odot I$; EG is a perfect line because G is the center of the figure consisting of 4 circles $\odot A, \odot C, \odot I$, and $\odot K$, and E is the center of $\odot E$.

Therefore points A, B, C, D, E, G, H, I, J, K all belong to some perfect lines, so the desired answer is point F.

Note: As the last diagram shows, we can prove that point F doesn't belong to any perfect lines: the area of the 5 circles above line EF is less than 2 and a half circles, and the area of the 5 circles above line FH is 3 circles, more than 2.5 circles. Thus, the line that passes through F and bisects the areas of the 5 circles must lie inside $\angle EFH$. But there is no red points in that region.